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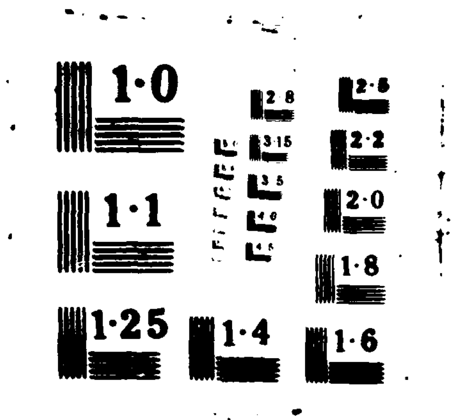
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ON THE CYCLABILITY OF
k-CONNECTED (k+1)-REGULAR GRAPHS

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ON THE CYCLABILITY OF

k-CONNECTED (k+l)-REGULAR GRAPHS

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1. Introduction

In the past fifteen years or so, there have been quite a number of papers dealing with variations on the following general theme. Given a graph G and a positive integer m, $m \leq |V(G)|$, find non-trivial conditions on G which will guarantee that given a set $S = \{v_1, ..., v_m\} \subseteq V(G)$, there exists a cycle C_S containing S. In the special case $m = |V(G)|$, we are dealing with conditions for the existence of Hamiltonian cycles, in itself a subject studied extensively by many graph theorists.

For the most recent survey of the subject for general m, the reader is directed to Holton [1983] and Plummer (1983). In particular, some interesting questions remain unsettled in the special case of regular graphs. Let C(m) denote the class of all graphs which have the property that every set of m points lie on some cycle. The largest m for which $G \in C(m)$ is called the cyclability of G. Now suppose $k \geq 3$ and let f(k) denote the largest integer j such that in every k-connected k-regular graph every j points lie on some cycle. It was proved by Holton (1982) and independently by Kelmans and Lomonosov (1982a) that $f(k) \geq k + 4$. This lower bound for f(k) is not believed to be best possible. For example, Holton, McKay, Plummer and Thomassen (1984) proved that $f(3) = 9$. This result was also obtained by Kelmans and Lomonosov independently and announced without proof in (1982a). Meredith (1973) constructed an infinite family of graphs which show, among other things, that $F(k) \geq 10k - 11$. Thus a rather large gap in possible values for f(k) remains at this writing. Recently, McQuaig and Rosenfeld (1984) have shown that for all even $k \geq 4$, there are infinite families of k-connected k-regular graphs with cyclabilities $8k - 4$ when $k \equiv 0 \pmod 4$ and $8k - 5$ when $k \equiv 2 \pmod 4$.

More recently, interest has been generated in the related question of cyclability of k-connected r-regular graphs for $r \geq k + 1$. First of all, Dirac (1960) proved that for any k-connected graph, regular or not, the cyclability is at least k. It is interesting to note that in the case of k-connected (k + 1)-regular graphs having k even, the Dirac bound cannot be improved. To see this, consider the complete bipartite graph $K_{k,k+1}$ where the bipartition sets U and W have $|U| = k$ and $|W| = k + 1$ respectively. The cyclability of $K_{k,k+1}$ is clearly k. We can easily modify $K_{k,k+1}$ to yield a graph H_k which is k-connected and (k + 1)-regular by replacing each point of W by a copy of the graph obtained from K_{k+2} by deleting a matching of cardinality $\frac{k}{2}$. (Figure 1.1 shows how this is done for $k = 4$.)

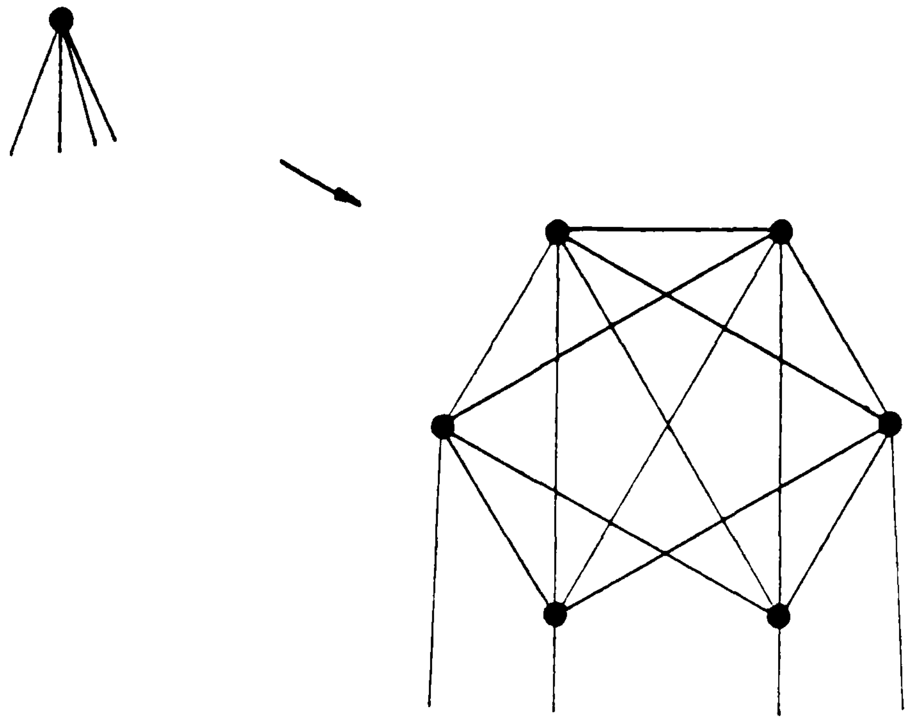


Figure 1.1

More generally, with k still even, but $r \geq k+1$, Holton (1982) has constructed other graphs which are k-connected and r-regular, but which do not lie in $C(k+1)$ and hence have cyclability precisely k.

Now suppose k is odd. If $r \geq k+2$, Holton (1982) has constructed k-connected r-regular graphs which do not lie in $C(k+1)$. This, again together with Dirac's bound, shows that any k-connected r-regular graph has cyclability = k, as long as k is odd and $r \geq k+2$.

So in a sense, the only case left unsettled here is that of k-connected $(k+1)$-regular graphs for $k \geq 3$ and k odd.

One can do a bit better than the Dirac bound here as was shown by Holton (1982), and independently by Kelmans and Lomonosov (1982b), via the following result.

Theorem 1.1. In any k-connected $(k+1)$-regular graph with $k \geq 3$ and odd, any $k+2$ points lie on a cycle.

Thus the cycability of such graphs is bounded below by $k+2$.

In fact, Kelmans and Lomonosov (1982b) claimed that the conclusion of Theorem 11 can be improved to $k+3$, but this claim is false, at least for $k=3$. For a counterexample due to the present authors, see Holton, (1983). Since Kelmans and Lomonosov did not publish the proof of the $k+3$ bound, the situation

for k odd and $k \geq 5$ is presently unknown, at least to the present authors. In his 1982 paper, Holton goes on to show that if k is odd and $k \geq 3$ and if $h(k)$ is the largest positive integer m for which all k-connected $(k+1)$-regular graphs lie in $C(m)$, then $h(k) \leq 9k$.

In the present paper, we will prove that, in fact, $h(k) \leq 2k - 1$. (This result was announced without proof by Holton (1983).) To accomplish this, we shall construct, given $k \geq 3$ and odd, a graph G_k which is k-connected and $(k+1)$-regular, but which has a set of $2k$ points which do not lie on a common cycle. The procedure will be as follows. First we construct a graph G'_k which is k-connected and which has all points with degree either k or $k+1$, but which has a set of $2k$ points not lying on any cycle. Then we modify G'_k first to obtain an intermediate graph G''_k and then, in turn, modify G''_k to obtain a k-connected $(k+1)$-regular graph G_k having a set of $2k$ points which lie on no common cycle.

The construction is done in two slightly different ways depending upon whether $k \equiv 1 \pmod 4$ or $k \equiv 3 \pmod 4$. The reader is encouraged to refer to graphs G'_5 and G'_7 to help understand the constructions in general. (See Figure 2.1.)

2. The Construction of G'_k.

Let $k \geq 3$ be an odd integer. In all cases G'_k will be a bipartite graph with bipartition $(X \cup Y \cup Y', Z \cup Z')$ where

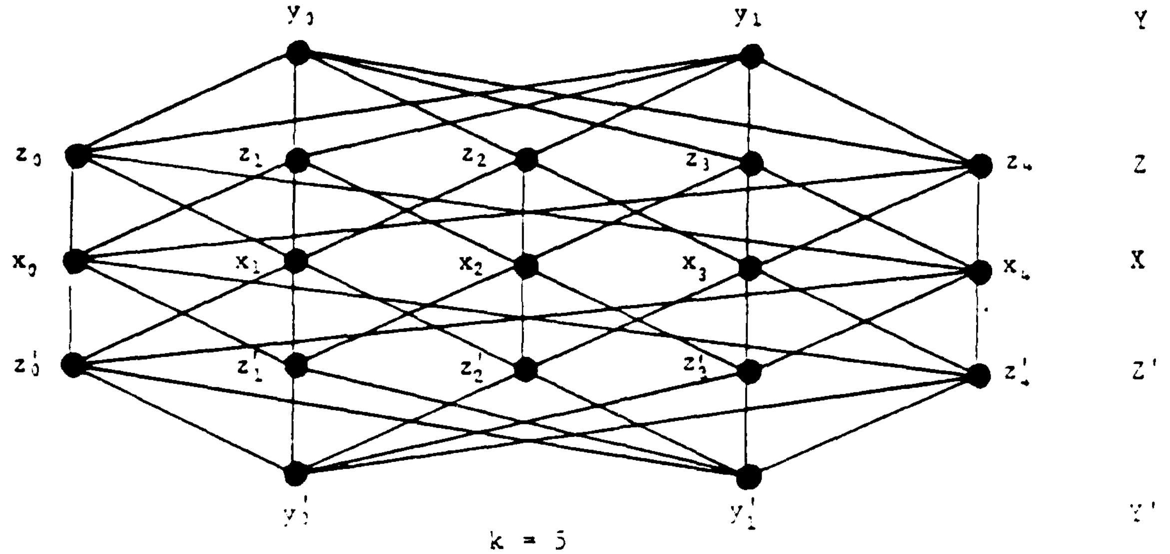
$$X = \{x_0, x_1, \ldots, x_{k-1}\},$$

$$Y = \{y_0, \ldots, y_{\frac{k-3}{2}}\}, \qquad Y' = \{y'_0, \ldots, y'_{\frac{k-3}{2}}\},$$

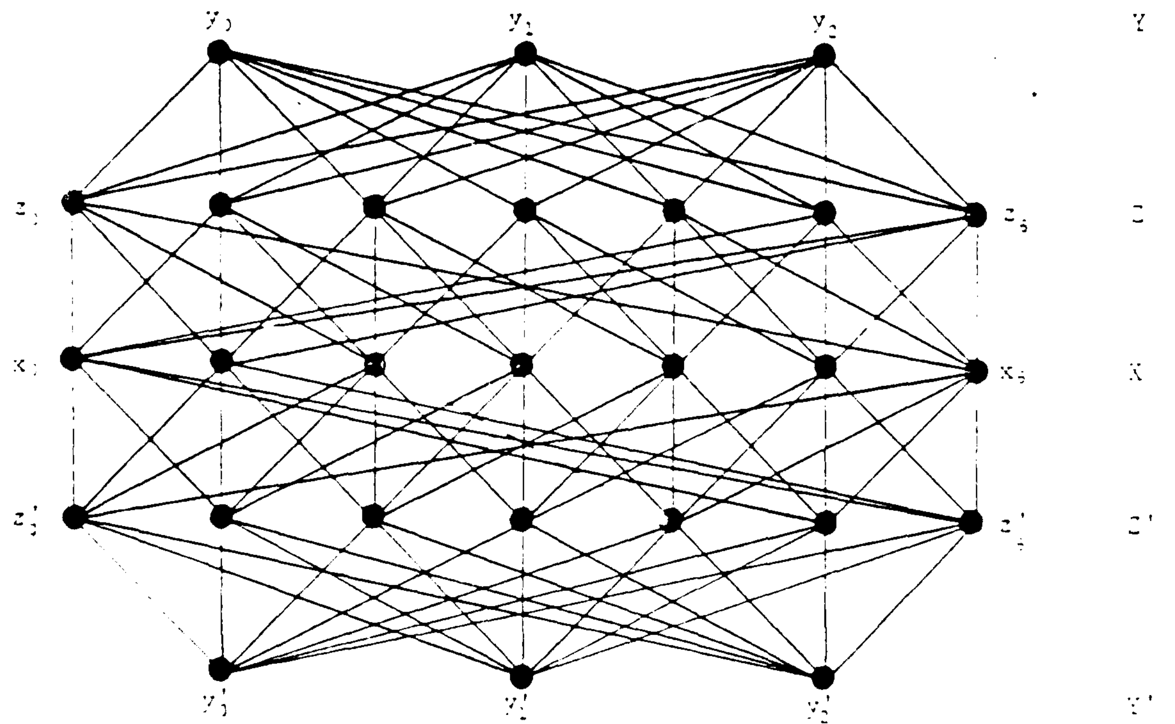
$$Z = \{z_0, \ldots, z_{k-1}\}, \qquad Z' = \{z'_0, \ldots, z'_{k-1}\}.$$

The lines in G'_k are defined as follows. Every point of Y (respectively Y') is adjacent to every point in Z (respectively Z'). For the remaining adjacencies we split the description into two cases. Suppose $k \equiv 1 \pmod 4$. For each $i = 0, \ldots, k-1$, both z_i and z'_i are adjacent to $x_i, x_{i-1}, x_{i+1}, \ldots, x_{i-\frac{k-1}{4}}, x_{i+\frac{k-1}{4}}$ where subscripts are taken modulo k. In the case in which $k \equiv 3 \pmod 4$, for $i = 0, \ldots, k-1$, both z_i and z'_i are adjacent to $x_i, x_{i-1}, x_{i+1}, \ldots, x_{i-\frac{k-3}{4}}, x_{i+\frac{k+1}{4}}$, where again the subscripts are taken modulo k.

The modulo k "circular symmetry" for adjacencies among the x_i's, z_j's and z'_k's is important to bear in mind and will prove to drastically reduce the number of cases we will have to treat in order to prove that G'_k is k-connected.



$k = 5$



$k = 7$

Figure 2.1

Finally, we note that G'_3 is just the well-known Hershel graph.

3. <u>The connectivity of G'_k</u>.

Note that in G'_k we have $\deg u = k$ for $u \in \cup Z \cup Z'$ and $\deg u = k+1$ for $u \in X$.

We now proceed to prove that G'_k is k-connected. To this end, let u and v be two distinct points in $V(G)$. We must find k openly disjoint paths joining u and v. We shall refer to such a family of paths as openly disjoint u - v paths. Here <u>openly disjoint</u> (hereafter abbreviated as <u>o.d.</u>) means that the paths joining u and v are otherwise pairwise point disjoint. We shall often refer to a set of k openly disjoint u - v paths as a <u>k-skein</u> joining u and v (or as a <u>u - v k-skein</u>).

1. First suppose $\{u,v\} \subseteq Y$. Say $u = y_0$ and $v = y_1$. Then $y_0z_0y_1, y_0z_1y_1, \ldots, y_0z_{k-1}y_1\}$ suffices as the u - v k-skein. The k-connection between two points of Y' follows by symmetry.

2. Suppose $u \in Y$ and $v \in Y'$. Without loss of generality, assume $u = y_0$ and $v = y'_0$. Then $\{y_0z_0x_0z'_0y'_0, \ldots, y_0z_{k-1}x_{k-1}z'_{k-1}y'_0\}$ suffices.

For the rest of the cases, we will treat the congruence classes $k \equiv 1 \pmod 4$ and $k \equiv 3 \pmod 4$ separately.

First suppose $k \equiv 1 \pmod 4$. (Thus $k \geq 5$.)

3a. Suppose $u \in Y$ and $v \in Z$, say $u = y_0$ and $v = z_{\frac{k-1}{2}}$. Note that for $i = 0, 1, \ldots, \frac{k-3}{2}$, we have z_i adjacent to $x_{\frac{k-1}{4}+i}$. So let

$$P_i = y_0\, z_i\, x_{\frac{k-1}{4}+i}\, z_{\frac{k-1}{2}}, \qquad \text{for } i = 0, \ldots, \tfrac{k-3}{2},$$

$$Q_i = y_0\, z_{\frac{k-1}{2}+i}\, y_i\, z_{\frac{k-1}{2}}, \qquad \text{for } i = 1, \ldots, \tfrac{k-3}{2} \qquad \text{and let}$$

$$R_1 = y_0\, z_{\frac{k-1}{2}} \quad \text{and} \quad S_1 = y_0\, z_{k-1}\, x_{\frac{3k-3}{4}}\, z_{\frac{k-1}{2}}.$$

Then $\{P_0, \ldots, P_{\frac{k-3}{2}}, Q_1, \ldots, Q_{\frac{k-1}{2}}, R_1, S_1\}$ is a u - v k-skein.

4a. Suppose $u \in Y$ and $v \in X$. Without loss of generality, suppose $u = y_0$ and $v = x_{\frac{k-1}{2}}$.

Let $P_i = y_0\, z_{\frac{k-1}{4}+i}\, x_{\frac{k-1}{2}}$, for $i = 0, \ldots, \frac{k-1}{2}$.

Now let $Q_i = y_0\, z_i\, x_{\frac{k-1}{4}+i}\, z'_{\frac{k-1}{4}+i}\, x_{\frac{k-1}{2}}$, for $i = 0, \ldots, \frac{k-5}{4}$

and let the "mirror images" of the Q_i's about the axis $z_{\frac{k-1}{2}}\, x_{\frac{k-1}{2}}\, z'_{\frac{k-1}{2}}$ be

$$R_i = x_{\frac{k-1}{2}}\, z'_i\, x_i\, z_{\frac{k-1}{4}+i}\, y_0, \quad \text{for } i = \frac{k+1}{2}, \ldots, \frac{3k-3}{4}.$$

We then have a total of $\frac{k+1}{2} + \frac{k-1}{4} + \frac{3k-3}{2} - \frac{k-1}{2} = k$ o.d. u - v paths as desired.

5a. Suppose $u \in Y$ and $u \in Z'$. Without loss of generality, let $u = y_0$ and $v = z'_{\frac{k-1}{2}}$. Then let

$$P_i = y_0\, z_{\frac{k-1}{4}+i}\, x_{\frac{k-1}{4}+i}\, z'_{\frac{k-1}{2}}, \quad i = 0, \ldots, \frac{k-1}{2},$$

$$Q_i = y_0\, z_i\, x_i\, z'_i\, y_i\, z'_{\frac{k-1}{2}}, \quad i = 0, \ldots, \frac{k-5}{4}, \quad \text{and let}$$

$$R_i = y_0\, z_{\frac{3k+1}{4}+i}\, x_{\frac{3k+1}{4}+i}\, z'_{\frac{3k+1}{4}+i}\, y'_{\frac{k-1}{4}+i}\, z'_{\frac{k-1}{2}} \quad \text{for } i = 0, \ldots, \frac{k-5}{4}.$$

We then have a total of $\frac{k+1}{2} + \frac{k-1}{4} + \frac{k-1}{4} = k$ o.d. u - v paths as sought.

6a. Suppose u and v are both in Z.

First note that any pair of z_i's have at least one common neighbour in X (and in fact, there are pairs of z_i's which have exactly one common neighbour). For example, (and for the sake of symmetry when working with the drawing in this case) let $u = z_{\frac{k-1}{4}}$ and let $v = z_{\frac{3k-3}{4}}$. Now let

$$P_i = z_{\frac{k-1}{4}}\, y_i\, z_{\frac{3k-3}{4}}, \quad i = 0, \ldots, \frac{k-3}{2} \quad \text{and let}$$

$$P_{\frac{k-1}{2}} = z_{\frac{k-1}{4}}\, x_{\frac{k-1}{4}}\, z_{\frac{3k-3}{4}}. \quad \text{Next let}$$

$$Q_i = z_{\frac{k-1}{4}}\, x_i\, z'_i\, y_i\, z'_{\frac{k+1}{2}+i}\, x_{\frac{k+1}{2}+i}\, z_{\frac{3k-3}{4}} \quad \text{for } i = 0, \ldots, \frac{k-3}{2}.$$

Then we have a total of $\frac{k-1}{2}+1+\frac{k-1}{2}=k$ o.d. $u-v$ paths as desired.

Now if the two z_i's chosen for u and v have $r \geq 2$ common neighbours in X, then in addition to the $\frac{k+1}{2}$ paths of type P_i above we get $r-1$ more of the form ux_jv. Taking these together with $\frac{k-1}{2}-(r-1)$ of type Q_i above, we get a total of $\frac{k+1}{2}+r-1+\frac{k-1}{2}-(r-l)=k$ o.d. $u-v$ paths as desired.

7a. Suppose $u \in Z$ and $v \in X$. Without loss of generality, assume $u = z_{\frac{k-1}{2}}$. There are now two cases to consider.

First suppose that $v \notin \Gamma(u) \cap X = \Gamma(z_{\frac{k-1}{2}}) \cap X$. (Here and throughout the rest of this paper $\Gamma(u)$ denotes the neighbourhood of u.) Let M denote the "vertical" matching of $\Gamma(z_{\frac{k-1}{2}})$ into all lines of which are of the form $x_i z'_i$. Then $|M| = \frac{k+1}{2}$ and we can find $\frac{k+1}{2}$ o.d. $u-v$ paths using M, r of them of length 3 where r is the number of neighbours of v in Z' which are covered by M and $\frac{k+1}{2}-r$ of length 5 which are of the form $v z'_j y'_k z'_m x_m z_{\frac{k-1}{2}}$, where $z'_m x_m \in M$, but $z'_m \notin \Gamma(v) \cap Z'$. On the other hand, v always has at least $\frac{k+1}{2}-1=\frac{k-1}{2}$ neighbours in Z which are not equal to $z_{\frac{k-1}{2}}$ and these can be used to form an additional $\frac{k-1}{2}$ o.d. $u-v$ paths. Again, then, we get $\frac{k+1}{2}+\frac{k-1}{2}=k$ o.d. $u-v$ paths as required.

So we may suppose that $v \in \Gamma(u) \cap X = \Gamma(z_{\frac{k-1}{2}}) \cap X$. This time we have $\frac{k-1}{2}$ o.d. $u-v$ paths of length 3 of form $z_{\frac{k-1}{2}} y_i z_j v$, the line $z_{\frac{k-1}{2}} v$ and $\frac{k-1}{2}$ additional paths of length 3 or 5 obtained as follows. Consider the matching M' of $\Gamma(u) \cap X$ "vertically" into Z'; that is, all lines of M' are of the form $x_i z'_i$. Delete from M' the line covering v and denote by M'' the resulting matching of size $\frac{k-1}{2}$. Now if M'' covers a neighbour of v we get a path of length 3, while if a line e of M'' does not cover a point of $\Gamma(v) \cap Z'$, we can find a $u-v$ path of length 5 using e by detouring through Y'.

Now if u and v have $r \geq 2$ common neighbours, then it is easy to see that there is still a set of $\frac{k-1}{2}$ $u-v$ paths of length 2 or 4 where the paths of length 4 are of the form $z_{\frac{k-1}{2}} y_j z_m x_m z'_i$, where $z_{\frac{k-1}{2}} = u$ and $z'_i = v$. Then one can find an additional $\frac{k+1}{2}-r$ $u-v$ paths of length 4 of the form $z_{\frac{k-1}{2}} x_m z'_m y_j z'_i$ and the remaining $k-\left(r+\frac{k+1}{2}-r+\frac{k+1}{2}-r\right)=r-1$ paths of length 6 having the form

$z_{\frac{k-1}{2}}\, y_j\, z_m\, x_m\, z'_m\, y'_n\, z_i$.

8a. Suppose $u \in Z$ and $v \in Z'$. Without loss of generality, let $u = z_{\frac{k-1}{2}}$. Now note that regardless of where v is in set Z', $r = |(\Gamma(u) \cap X) \cap (\Gamma(v) \cap X)| > 0$. So let $s_1 = |(\Gamma(u) \cap X) - (\Gamma(v) \cap X)|$, let $s_2 = |(\Gamma(u) \cap X) - (\Gamma(v) \cap X)|$ and $s_3 = |X - (\Gamma(u) \cup \Gamma(v))|$. Then clearly $r + s_1 + s_2 + s_3 = k$. Those members of X counted by r give rise to r o.d. $u - v$ paths of length 2. For each point x_i of X counted by s_1 take line $x_i z'_i$, for each counted by s_2 take line $x_i z_i$ and for each counted by s_3 take the path $z_i\, x_i\, z'_i$. The lines $x_i\, z'_i$ counted by s_1 give rise to $u - v$ paths of length 4 all of form $u\, x_i\, z'_i\, y'_j\, v$, those counted by s_2 yield $u - v$ paths of length 4 of form $u\, y_j\, z_i\, x_i\, v$ and those counted by s_3 yield $u - v$ paths of length 6 all of the form $u\, y_j\, z_i\, x_i\, z'_i\, y_j\, v$. Altogether, these form a collection of k o.d. $u - v$ paths.

9a. Finally, suppose both u and $v \in X$. Let $u = x_i$ and $v = x_j$. Consider $\Gamma(x_i) \cap Z = N_Z(x_i)$. If $z_m \in N_Z(x_i)$ then if it is also in $\Gamma(x_j) \cap Z = N_Z(x_j)$ we have a path of length 2 - namely $x_i\, z_m\, x_j$ - joining x_i and x_j. On the other hand, if $Z_m \in N_Z(x_i - N_Z(x_j)$ then we have a path of length 4 - namely $x_i\, z_m\, y_n\, z_s\, x_j$ - joining x_i and x_j. This yields a total of $\frac{k+1}{2}$ o.d. $u - v$ paths and they all lie within $G'_k[X \cup Z \cup Y]$. But clearly there is a second set of $\frac{k+1}{2}$ o.d. $u - v$ paths (the reflections of the first set of paths in the X axis) which, together with the first set yields a total of $k + 1$ o.d. $u - v$ paths as sought.

Now let us suppose $k \equiv 3 \pmod 4$.

3b. Suppose $u \in Y$ and $v \in Z$. Without loss of generality, suppose $u = y_0$ and $v = z_{\frac{k-1}{2}}$.

Let $P_i = y_0\, z_i\, x_{\frac{k+1}{4}+i}\, z_{\frac{k-1}{2}}$, for $i = 0, \ldots, \frac{k-3}{2}$,

$Q_i = y_0\, z_{\frac{k-1}{2}+i}\, y_i\, z_{\frac{k-1}{2}}$, for $i = 1, \ldots, \frac{k-3}{2}$,

$R_1 = y_0\, z_{\frac{k-1}{2}}$ and $S_1 = y_0\, z_{k-1}\, x_{\frac{3k-1}{4}}\, z_{\frac{k-1}{2}}$.

Then $\{P_0 \ldots, P_{\frac{k-3}{2}}, Q_1, \ldots, Q_{\frac{k-3}{2}}, R_1, S_1\}$ is a k-skein joining u and v.

4b. Suppose $u \in Y$ and $v \in X$. Without loss of generality, suppose $u = y_0$ and $v = x_{\frac{k-1}{2}}$.

Let $P_i = y_0 z_{\frac{k-3}{4}+i} x_{\frac{k-1}{2}}$, $i = 0, \ldots, \frac{k-1}{2}$. Now if $k \neq 3$, let

$Q_i = y_0 z_i x_{\frac{k+1}{4}+i} z'_{\frac{k+1}{4}+i} x_{\frac{k-1}{2}}$, $i = 0, \ldots, \frac{k-7}{4}$ and if $k = 3$, let $Q_0 = \phi$.

Also let $R_i = x_{\frac{k-1}{2}} z'_i x_{i+1} z_{\frac{k+1}{4}+i} y_0$, for $i = \frac{k-1}{2}, \ldots, \frac{3k-5}{4}$. We then have

a total of $\frac{k+1}{2} + \frac{k-3}{4} + \frac{3k-5}{4} - \frac{k-3}{2} = k$ o.d. $u-v$ lines in all cases.

5b. Suppose $u \in Y$ and $v \in Z'$. Without loss of generality, let $u = y_0$ and $v = z'_{\frac{k-1}{2}}$.

Then let $P_i = y_0 z_{\frac{k+1}{4}+i} x_{\frac{k+1}{4}+i} z'_{\frac{k-1}{2}}$ for $i = 0, \ldots, \frac{k-1}{2}$

$Q_i = y_0 z_i x_i z'_i y'_i z'_{\frac{k-1}{2}}$ for $i = 0, \ldots, \frac{k-3}{4}$, and

and let $R_i = y_0 z_{\frac{3k+3}{4}+i} x_{\frac{3k+3}{4}+i} z'_{\frac{3k+3}{4}+i} y'_{\frac{k+1}{4}+i} z'_{\frac{k-1}{2}}$ for $i = 0, \ldots, \frac{k-7}{4}$, when $k \geq 7$.

(For $k = 3$, let $R_3 = \phi$.)

We then have a total of $\frac{k+1}{2} + \frac{k+1}{4} + \frac{k-3}{4} = k$ o.d. $u-v$ paths as desired.

6b. Suppose u and v are both in Z. Again, as in 6a, note that every pair of x_i's have at least one common neighbour in X and in fact there are pairs with exactly common neighbour. Let $u = z_{\frac{k-3}{4}}$ and $v = z_{\frac{3k-5}{4}}$, for example.

First suppose $k \geq 7$.

Now let

$P_i = z_{\frac{k-3}{4}} y_i z_{\frac{3k-5}{4}}$, $i = 0, \ldots, \frac{k-3}{2}$ and let

$P_{\frac{k-1}{2}} = z_{\frac{k-1}{4}} x_{\frac{k-1}{2}} z_{\frac{3k-3}{4}}$. Next let

$Q_i = z_{\frac{k-3}{4}} x_i z'_i y_i z'_{\frac{k+1}{2}+i} x_{\frac{k+1}{2}+i} z_{\frac{3k-5}{4}}$ for $i = 0, \ldots, \frac{k-3}{2}$.

Thus we obtain a total of $\frac{k-1}{2}+1 + \frac{k-1}{2} = k$ o.d. $u - v$ paths as desired. If $k = 3$, then 3 o.d. $u - v$ paths are obvious.

Now if $k \geq 7$ and the 2 z_i 's chosen for u and v have $r \geq 2$ common neighbours in X, then in addition to the $\frac{k+1}{2}$ paths of type P_i above, we get $r - 1$ more of the form $u\, x_j\, v$. Taking these together with $\frac{k-1}{2} - (r-1)$ of type Q_i above, we get a total of $\frac{k+1}{2} + r - 1 + \frac{k-1}{2} - (r-1) = k$ o.d. $u - v$ paths as desired.

The proofs of Cases 7b ($u \in Z$, $v \in X$), 8b ($u \in Z$, $v \in Z'$) and 9b ($u, v \in X$) are identical to those for Cases 7a, 8a and 9a respectively.

This completes the proof that G'_k is k-connected.

4. The Construction of G_k.

Recall that in graph G'_k each point in $Y \cup Y' \cup Z \cup Z'$ has degree k, while each point in X has degree $k+1$. We now proceed to construct a $(k+1)$-regular graph G_k from G'_k as follows.

First consider each line joining some $y_i \in Y$ to a $z_j \in Z$. Insert a new "midpoint" on this line and call it α_{ij}. Similarly, insert a midpoint β_{ij} on each line joining a z_i to an x_j. Midpoints are similarly inserted on lines joining a y'_i to a z'_j and on lines joining a z'_i to an x_j. They are called α'_{ij} and β'_{ij} respectively.

Now we replace each point of $Y \cup Y' \cup Z \cup Z'$ with a set of points as follows.

First suppose $k \equiv 1 \pmod 4$. For each $i \in \{0, \dots, \frac{k-5}{2}\}$, replace y_i by a set A_i of $2k$ new points joined two by two to midpoints $\alpha_{i0}, \alpha_{i1}, \dots, \alpha_{i(k-1)}$ respectively. Now replace $y_{\frac{k-3}{2}}$ by a set $B_{\frac{k-3}{2}}$ consisting of $2k$ points, k of them joined one at a time to each $\alpha_{\frac{k-3}{2}j}$ for $j = 0, \dots, k-1$ and the remaining k joined to yet another new point b. Replace the y'_i 's with sets A'_i and $B'_{\frac{k-3}{2}}$ in a symmetric manner.

Next, replace each $z_j \in Z$ with a set C_j of points as follows. For each line of the form $\alpha_{ij} z_j$ for $i \neq \frac{k-3}{2}$ insert $k-1$ new points into C_j and join each to α_{ij}. Also replace $\alpha_{\frac{k-3}{2}j} z_j$ with an additional k new points. Furthermore, for each line of the form $\beta_{jr} x_r$, insert k new points into C_j and join each to β_{jr}. (See Figure 4.Ia.) Thus altogether, C_j contains $\frac{(k-3)}{2}(k-1) + k + \frac{(k+1)}{2}k = \frac{2k^2 - k + 3}{2}$ points, which since $k \equiv 1 \pmod 4$ is an even number.

Thus when $k \equiv 1 \pmod 4$, all of the sets A_i, C_j and $B_{\frac{k-3}{2}}$ contain an even number of points.

"Mirror image" sets A'_i, $B'_{\frac{k-3}{2}}$, C'_j and point b' are constructed analogously.

Now since each of the sets A_i, A'_i, $B_{\frac{k-3}{2}}$, $B'_{\frac{k-3}{2}}$, C_j, C'_j have more than k points and each is even, we may invoke Lemma 4a of Wang and Kleitman (1973) to conclude that there exists a k-connected k-regular graph on each of these sets of points. Insert such a k-regular graph on each such point set. Finally, join points b and b'. Clearly, the resulting graph G_k is $(k+1)$-regular.

Now suppose $k \equiv 3 \pmod 4$. In this case, we can construct a $(k+1)$-regular G'_k which is even simpler than that built for the case $k \equiv 1 \pmod 4$ in that no "special" replacement for $y_{\frac{k-3}{2}}$ is necessary.

In G'_k, insert midpoints α_{ij} and β_{ij} as before. For each $i = 1, \ldots, \frac{k-3}{2}$ replace y_i by a set A_i of $2k$ points joined two by two to each α_{ij}. Replace each z_j by a set C_j consisting of $\frac{k-1}{2}(k-1)$ joined $k-1$ at a time to each of the $\frac{k-1}{2}$ different α_{ij}'s. Also add $k\frac{k-1}{2}$ additional points to C_j joined k at a time to each of the different β_{ij}'s. (See Figure 4.Ib.)

Once again construct the "mirror image" sets A'_i and C'_j analogously.

Now each A_i and A'_i contains $2k$ points while each C_j and C'_j contains $\frac{(k-1)^2}{2} + \frac{k(k-1)}{2} = \frac{1}{2}(2k^2 - 3k + 1)$ points which is also an even number since $k \equiv 3 \pmod 4$. Thus again by the Wang and Kleitman result we can construct k-connected k-regular graphs on each of these sets and hence obtain our $(k+1)$-regular graph G'_k.

$k \equiv 1 (\mathrm{mod}\ 4)$:

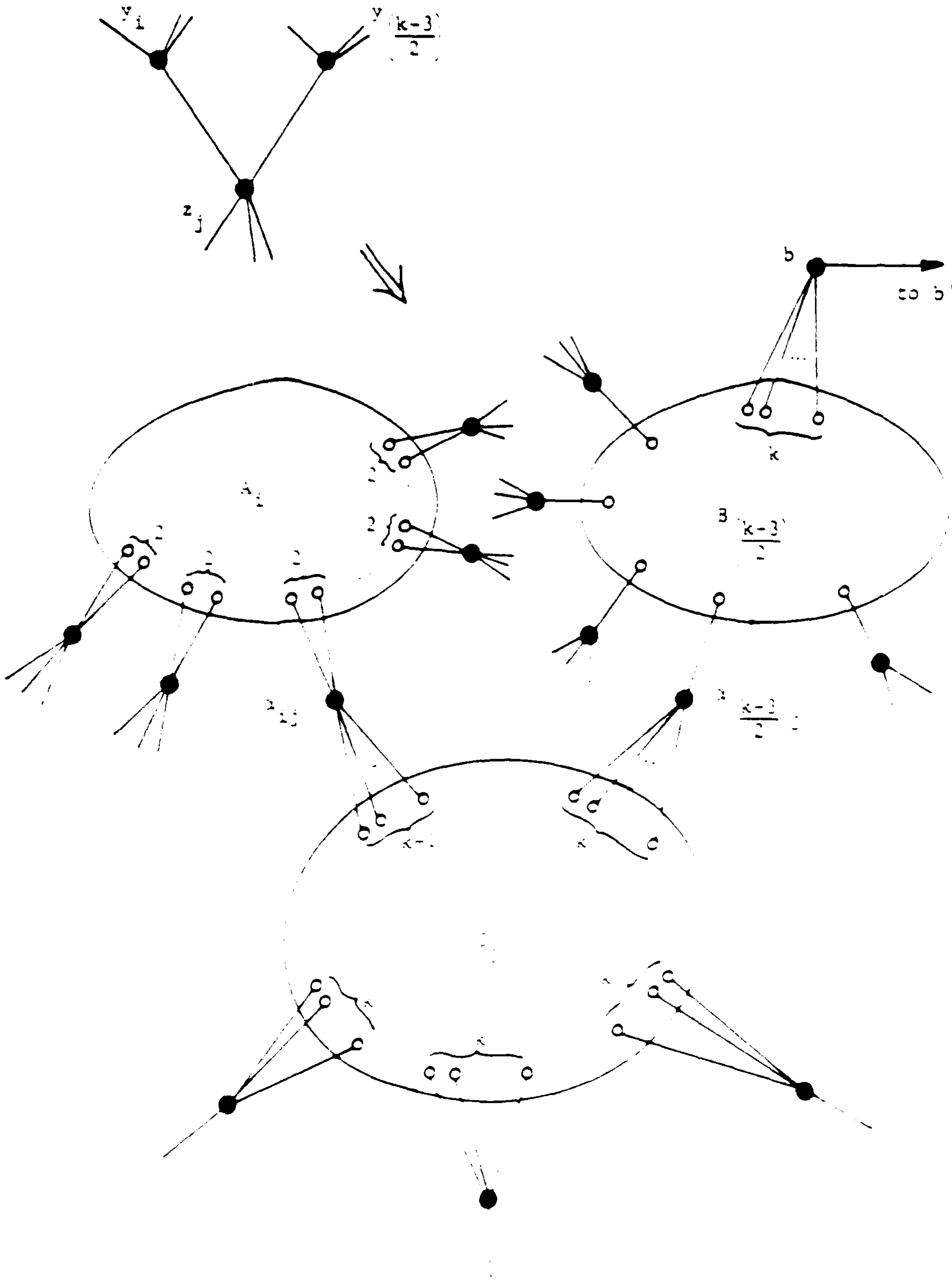


Figure 4 (a)

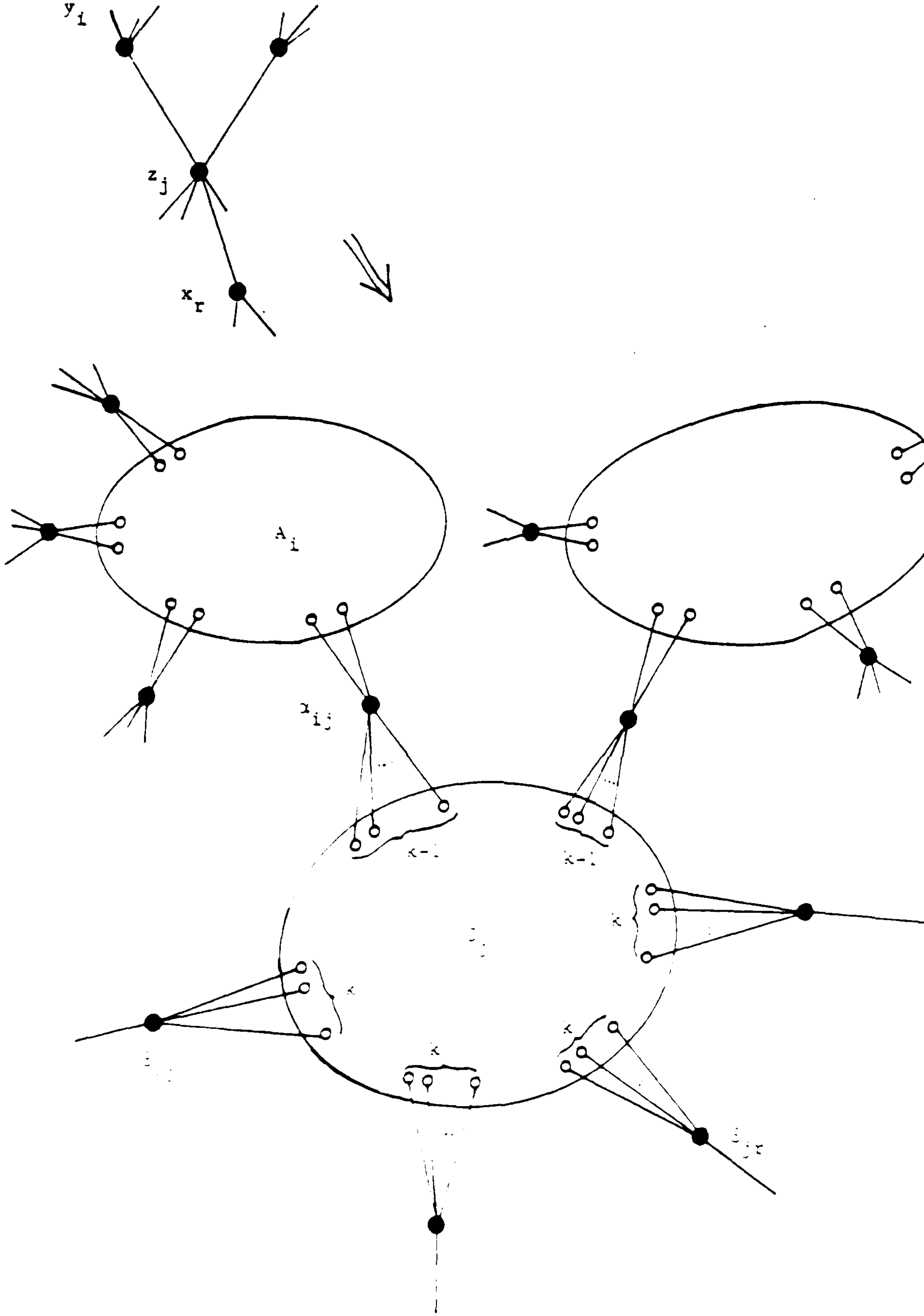


Figure 4.1 (b)

5. The Connectivity of G_k.

To prove that G_k is k-connected we proceed in two steps. First we consider an intermediate graph G''_k obtained from G'_k by inserting only the C_j's. (From this point on we shall denote the subgraphs guaranteed by the Wang and Kleitman result on A_i by $\langle A_i \rangle$, on $B_{\frac{k-3}{2}}$ by $\langle B_{\frac{k-3}{2}} \rangle$, etc.)

We now proceed to show G''_k to be k-connected. Let u and v be two distinct points in G''_k.

Suppose first that neither u nor v is a midpoint.

1. If u and v lie in the same C_j then there exist k o.d. u - v paths in $\langle C_j \rangle$, since $\langle C_j \rangle$ is k-connected. The analogous result holds when u and v lie in the same C'_j.

2. If u and v lie in two different C_j's, C'_j's or one in a C_j and the other in a C'_j, then there exist k o.d. u - v paths since such a set of paths exists in G'_k. More precisely, suppose $u \in C_u$ and $v \in C_v$. In C_u for each midpoint adjacent to C_u choose a point in C_u different from u. (Henceforth we shall refer to such a point as a foot of this midpoint in C_u.) This is possible because each midpoint has at least $k - 1 > 2$ such feet in C_u. So the feet selected in this way form a set of k distinct points in C_u different from u. Now since $\langle C_u \rangle$ is k-connected by a well-known variation of Menger's Theorem, there exists a fan of paths in $\langle C_u \rangle$ from u to each of the k feet chosen.

Repeat this procedure in $\langle C_v \rangle$ and use these two k-fans, together with suitable pieces of the k o.d. paths in G'_k joining $\langle C_u \rangle$ contracted to a point to $\langle C_v \rangle$ contracted to a point.

This argument is also valid if u and v are in the same C_j, the same C'_j or one is in a C_j and the other in a C'_j.

3. Suppose $u \in Y \cup Y' \cup X$ and $v \in C_j$ or C'_j. Without loss of generality, suppose $v \in C_j$. Since G'_k is k-connected, there exist k o.d. u - v paths in G'_k and using the argument of Case 2, we can find k o.d. u - v paths in G''_k.

4. If $\{u,v\} \subseteq Y \cup Y' \cup X$, then k o.d. u - v paths are found using the k-connectedness of G'_k and the fact that all $\langle C_j \rangle$'s are themselves k-connected.

So it remains to treat the cases when at least one of u and v is a midpoint. Note that in G''_k, the midpoints have degree k if they lie between Y and Z or between Y' and Z', and they have degree $k+1$ if they lie between X and Z or between X and Z'.

First suppose both u and v are midpoints in G''_k.

Let us now first consider the case when u and v are adjacent to the same $\langle C_i \rangle$ (or $\langle C'_j \rangle$). Then u and v are adjacent to at least k-1 different points of C_i respectively. By Menger's Theorem there are at least k-1 o.d. u - v paths in the subgraph $\langle C_i \cup \{u,v\} \rangle$ of G''_k. Call them $P_1, \ldots, P_{k-1}$. Also since G'_k is 2- connected, there is a cycle N in G'_k containing the lines L_u and L_v (whose midpoints are u and v) and hence $N - L_u - L_v$ is a path which may be used to construct a path Q joining u and v which is openly disjoint from all the P_i's. Thus $\{P_1, \ldots, P_{k-1}, Q\}$ is the desired u - v k-skein.

Now suppose u and v are adjacent (as midpoints) to different $\langle C_j \rangle$'s, say C_u and C_v respectively. Now in G'_k the 2 points corresponding to the contractions of $\langle C_u \rangle$ and $\langle C_v \rangle$ are joined by k o.d. paths. Call them $P_1, \ldots, P_k$. One of these - say P_1 - uses line L_u. Choose k-1 distinct feet of u in C_u. Call this set U_1. Also for each path P_i, $i \neq 1$, choose exactly one foot in C_u. Call this set U_2. We then have $U_1 \cup U_2 \subseteq C_u$, $U_1 \cap U_2 = \phi$ and $|U_1| = |U_2| = k-1$. Since $\langle C_u \rangle$ is (k-1)-connected, by Menger's Theorem there exist k-1 totally disjoint paths in $\langle C_u \rangle$ joining the points of U_1 to those of U_2. A similar argument applies to $\langle C_v \rangle$. Using these paths within $\langle C_u \rangle$ and $\langle C_v \rangle$ as well as paths $P_1, \ldots, P_k$, we can construct k o.d. u - v paths in G''_k.

Finally, suppose u is a midpoint in G''_k, but v is not. Suppose u is adjacent to $\langle C_u \rangle$. But this is even simpler than the proceding case. In G'_k, let $P_1, \ldots, P_k$ be k o.d. paths joining the contraction to a point of $\langle C_u \rangle$ with point v. As before, let U_1 be a set of k-1 feet of u in C_u and choose U_2 so that it contains precisely one foot of each of the rest of the midpoints adjacent to $\langle C_u \rangle$. Then $|U_1| = |U_2| =$ k-1, $U_1 \cap U_2 = \phi$ and since $\langle C_u \rangle$ is (k-1)-connected we can proceed as before to get k o.d. u - v paths.

This completes the proof that G''_k is k-connected.

Now insert the A_i's, A'_i's, (and $B_{\frac{k-3}{2}}$ and $B'_{\frac{k-3}{2}}$ if $k \equiv 1 \pmod 4$)) into G^*_k. Also insert points b and b' together with their respective k-fans to $B_{\frac{k-3}{2}}$ and $B'_{\frac{k-3}{2}}$. But do not join b and b' yet.

Actually, we will now show that $G_k - b - b'$ is k-connected. So suppose $\{u,v\} \subseteq V(G_k) - \{b,b'\}$.

I. Suppose $\{u,v\} \cap (A_i \cup A'_j) = \phi$, for all i,j and $\{u,v\} \cap (B_{\frac{k-3}{2}} \cup B'_{\frac{k-3}{2}}) = \phi$ when $k \equiv 1 \pmod 4$. Then since G^*_k is k-connected there are k o.d. u - v paths $P_1, \ldots, P_k$ in G^*_k. Since all $\langle A_i \rangle$'s, $\langle A'_j \rangle$'s, $B_{\frac{k-3}{2}}$ and $B'_{\frac{k-3}{2}}$ are connected, paths $P_1, \ldots, P_k$ give rise to k o.d. paths $Q_1, \ldots, Q_k$ joining u and v in $\underline{G_k}$.

Before proceeding to the next case, we state and prove the following statement.

<u>Claim</u> (a) If $y_i \in Y$ corresponds to inserted subgraph A_i (respectively $B_{\frac{k-3}{2}}$) in G_k and if $L_1, \ldots, L_k$ are the k lines incident with y_i in G'_k, then given any point $u \in A_i$ (respectively $B_{\frac{k-3}{2}}$) there exists a k-fan in A_i (respectively $B_{\frac{k-3}{2}}$) which can be extended to a k-fan joining u to the midpoints of $L_1, \ldots, L_k$.

(b) Analogous statements hold for $y'_i \in Y'$ with respect to A'_i (respectively $B'_{\frac{k-3}{2}}$).

<u>Proof of Claim</u>. We prove only part (a) as (b) is proved in just the same way. Suppose y_i corresponds to A_i. Choose any point $u \in A_i$. Then u is one of exactly two feet in A_i of some midpoint x_{ij}. Suppose w is the other of these two feet. Form a set U of k points by including w and exactly one of the two feet of all the other k-1 midpoints adjacent to A_i. Then since $u \notin U$ and $\langle A_i \rangle$ is k-connected, there exists a fan of paths from u to each of the k points in U which in turn leads to the k-fan sought.

Now suppose y_i corresponds to $B_{\frac{k-3}{2}}$. That is, $y_i = y_{\frac{k-3}{2}}$. Let $u \in B_{\frac{k-3}{2}}$. There are two cases to consider.

First suppose u is the foot of new point b. Then since u is not a foot of any of the midpoints of $L_1, \ldots, L_k$ and since $B_{\frac{k-3}{2}}$ is k-connected, there is a k-fan of paths from u to the (unique) foot of each L_i in $B_{\frac{k-3}{2}}$. There are k such feet and this fan clearly extends to one from u to each of the k midpoints of $L_1, \ldots, L_k$.

So suppose u is the foot of some L_j in $B_{\frac{k-3}{2}}$. Without loss of generality, suppose u is the foot of L_1. Then since $B_{\frac{k-3}{2}}$ is k-connected, there is a fan at u to the feet in $B_{\frac{k-3}{2}}$ of each of the $k - 1$ lines $L_2, \ldots, L_k$. These $k - 1$ paths, together with the line from the foot of L_1 to the midpoint of L_1, clearly extends to a fan from u to the midpoint of each of $L_1, \ldots, L_k$ as desired. This completes the proof of the Claim.

2. Now suppose at least one of u, v lies in an A_i, A'_i, $B_{\frac{k-3}{2}}$, or $B'_{\frac{k-3}{2}}$, but that u and v do not both lie in the same one of these sets. Since G^*_k is k-connected, there are k o.d. $u - v$ paths in G^*_k which together with the fans guaranteed by the above Claim, where necessary, yield k o.d. $u - v$ paths in G_k.

3. If both u and v lie in the same $\langle A_i \rangle$, $\langle A'_i \rangle$, $\langle B_{\frac{k-3}{2}} \rangle$ or $\langle B'_{\frac{k-3}{2}} \rangle$, then since all of these subgraphs are k-connected, there exist k o.d. $u - v$ paths as desired.

Thus $G_k - b - b'$ is k-connected. It remains now to add points b and b', join them to k points each in $B_{\frac{k-3}{2}}$ and $B'_{\frac{k-3}{2}}$ as described earlier. But if we join b to its k points, the resulting graph $G_k - b'$ is k-connected by Menger's Theorem and then joining b' to its k neighbours, the resulting graph $G_k - bb'$ is k-connected by the same reasoning. But then adding line bb' we obtain G_k which must be k-connected. Clearly G_k is $(k + 1)$-regular.

Finally, we note that trivially the $2k$ points of $Z \cup Z'$ lie on no common cycle in G_k since $Z \cup Z'$ is an independent set and $|V(G_k) - (Z \cup Z')| = |Y \cup Y' \cup X| = k - 1$.

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